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ROBIN THOMAS

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13. ABSTRACT (Maximum 200 words)

This report summarizes work performed under this contract. The main topics are: Hadwiger's conjecture, rooted subdivisions of  $K_4$ , results of PI's student D.P. Sanders, representativity of surface embeddings, the Four Color Theorem, Younger's conjecture.

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# GRAPH MINORS: STRUCTURE THEORY AND ALGORITHMS

Final report on project number N00014-93-1-0325

sponsored by the Office of Naval Research

This report covers the period from 1 February 1993 through 31 January 1996.

1. The case  $n = 6$  of Hadwiger's conjecture. This research was carried out prior to the commencement of the grant; a revised version of paper [1] was prepared.
2. Rooted subdivisions of  $K_4$ . With Neil Robertson, the PI worked on the following problem: Given a graph  $G$  and four vertices of  $G$ , when does there exist a  $K_4$ -subdivision with nodes ("branch-vertices") precisely the four given vertices? Applications of this range from pure graph theory (Dirac's conjecture, Hajos' conjecture, Kelmans' conjecture) to applied problems such as efficient call routing. Robertson and the PI formulated a conjecture about 5-connected planar graphs that was subsequently proved by Xingxing Yu.
3. Daniel P. Sanders has written (under my supervision) paper [2], where he proves a special case of a conjecture of Lovasz that in a 5-connected graph, any five independent edges are contained in a simple circuit, unless they form a cut.
4. Daniel P. Sanders has defended his thesis Linear time algorithms for tree-width four. His main result is a practical linear-time algorithm to find tree-decompositions of graphs of width at most four, which in turn implies that many (even NP-hard) problems can be solved in linear time for graphs of tree-width at most four.
5. Paul Seymour and I have improved the bound on "representativity" of a surface embedding that guarantees uniqueness. More precisely, let  $G$  be a graph embedded in a non-spherical surface  $S$  of genus  $g$  in such a way that every homotopically non-trivial closed curve in  $S$  meets the graph at least  $r$  times. Robertson and Vitray have shown that if  $r$  is at least  $2g + 3$  then there is no embedding in a surface of smaller genus. With Seymour we have improved this bound to  $c \log(g) / \log \log(g)$ .
6. Vojtech Rodl and I have shown that graphs not containing a subdivision of a fixed clique have bounded "arrangeability", a concept recently introduced by Chen and Schelp. This implies that graphs with no fixed clique subdivision have bounded "game chromatic number", and have "linearly bounded Ramsey numbers".
7. The Four Color Problem was first formulated by Francis Guthrie in 1852, and remained open until 1976 when Appel and Haken gave a proof. Since then it has been known as the Four Color Theorem. However, the proof of Appel and Haken is so complicated, that it has not and probably never will be independently verified. On the algorithmic side, while it gives a quartic algorithm to four-color planar graphs, the algorithm suffers from the same weaknesses as the proof itself. Neil Robertson, Daniel P. Sanders, Paul Seymour and I have found a much simpler proof. We use the same approach as Appel and Haken, that is, finding an unavoidable set of reducible configurations. While the reducibility part of the proof is very similar to Appel and Haken's, it is the unavoidability part where our methods differ. Our method enabled us to reduce the size of the unavoidable set

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from about 1528 to 633. Another advantage of our proof is that based on it we managed to design a quadratic algorithm to four-color planar graphs. This is an improvement over the quartic algorithm of Appel and Haken. We are currently implementing the algorithm, and believe that it will be a practical one.

The new proof of the Four Color Theorem opens up the possibility of finding efficient coloring algorithms for larger classes of graphs. There are several conjectures to this effect, of which the one that appears to be approachable is Tutte's 3-edge-coloring conjecture. The Four Color Theorem is equivalent to the statement that every 2-edge-connected cubic planar graph is 3-edge-colorable. Tutte conjectured in the mid 1960's that planar here can be replaced by the much weaker condition that  $G$  has no minor isomorphic to the Petersen graph. With Robertson and Seymour we managed to reduce this to two classes of graphs: apex graphs (graphs such that the deletion of some vertex makes them planar) and double-cross graphs (graphs that can be drawn in the plane with two crossings on the boundary of the infinite region). The hope is that our proof of the Four Color Theorem can be adapted to prove Tutte's conjecture for these two classes of graphs. I am currently working with Daniel P. Sanders on the apex case.

8. B. Reed, N. Robertson, P.D. Seymour and I managed to prove a conjecture of Younger that for every integer  $t$  there exists an integer  $f(t)$  such that every directed graph either has  $t$  disjoint directed circuits, or a set of at most  $f(t)$  vertices meeting all directed circuits. This immediately implies that a number of important problems can be solved in polynomial time when restricted to digraphs with bounded number of disjoint directed circuits. More importantly, in the course of proving this result we have defined a new concept, path-width of a digraph, an analogue of an already reasonably well-understood concept of path-width of undirected graphs. It appears that this concept will play the same role in digraph theory as tree-width does for undirected graphs. In particular, I expect that this notion will lead to the design of efficient algorithms as well as practical ones once its theory is sufficiently developed. This is a research project for a team of investigators for several years, and includes many problems suitable for graduate students.

Part of this work is being carried out in collaboration with Paul Seymour of Bellcore and Neil Robertson of Ohio State University. Some of the problems that motivate this research originated in the applied research division of Bellcore. A patent for the design of SONET networks based on concepts and ideas studied in this project (most notably the concept of tree-width of a graph) has been developed at Bellcore. A similar idea has also been successfully used in an algorithm to solve large traveling salesman problems by Applegate, Bixby, Cook and Chvatal. Using this algorithm they managed to solve a traveling salesman problem on over 4,000 vertices, the largest traveling salesman problem ever solved exactly. There have also been interactions with the University of Bordeaux, University of Paris VI and Bielefeld University.

### Publications

1. Hadwiger's conjecture for  $K_6$ -free graphs (with N. Robertson and P. D. Seymour), *Combinatorica* 13 (1993), 279-361.

2. Daniel P. Sanders, Circuits through five edges in 5-connected graphs, submitted.
3. Daniel P. Sanders, Linear-time algorithms for graphs of tree-width at most four, Ph.D. dissertation, Georgia Institute of Technology, 1993.
4. On the genus of a random graph (with V. Rödl), *Random Structures and Algorithms* 6 (1995), 1-12.
5. Arrangeability and clique subdivisions (with V. Rödl), *Mathematics of Paul Erdős*, R. L. Graham and J. Nešetřil eds.
6. Uniqueness of highly representative surface embeddings (with P. D. Seymour), *J. Graph Theory*.
7. A new proof of the four-color theorem (with N. Robertson, D. P. Sanders and P. D. Seymour), to appear in *Electronic Research Announcements of the AMS*.
8. 5-connected toroidal graphs are Hamiltonian (with X. Yu), submitted.
9. The four-colour theorem (with N. Robertson, D. P. Sanders and P. D. Seymour), submitted.
10. Packing directed circuits (with B. Reed, N. Robertson and P. D. Seymour), submitted.
11. Reducibility in the four-color theorem (with N. Robertson, D. P. Sanders and P. D. Seymour), available from <ftp://ftp.math.gatech.edu/pub/users/thomas/four>.
12. Discharging cartwheels (with N. Robertson, D. P. Sanders and P. D. Seymour), available from <ftp://ftp.math.gatech.edu/pub/users/thomas/four>.
13. Cyclically five-connected cubic graphs (with N. Robertson and P. D. Seymour) (to be submitted).
14. Excluded minors in cubic graphs (with N. Robertson and P. D. Seymour) (to be submitted).

#### Graduate students

- Daniel P. Sanders, Ph.D. 1993. Current position: Assistant Professor, Ohio State University
- Christopher Anhalt, Diplomarbeit (German MS thesis), 1994 (no ONR support)
- Matthias Hayer, MS thesis, 1994 (no ONR support)
- Christopher Carl Heckman, Research Assistant, 1994, 1995, 1996
- Kevin Haulk, Research Assistant, 1995
- Thomas Fowler, Research assistant 1993, 1994, 1995, 1996
- Zhiqing Lu (no ONR support)
- Barrett Walls, Research Assistant, 1995, 1996

#### Invited lectures

- Seminaire LABRI, University of Bordeaux, France, April 1993
- Graph Theory Seminar, University of Paris VI, April 1993
- SFB Seminar, Bielefeld University, May 1993 (2 lectures)
- Graph Theory Seminar, Charles University, Prague, Czech Republic, May 1993

- Sixth Cumberland Conference, Memphis, TN, May 1993
- AMS meeting, Dekalb, IL, May 1993
- Third China-USA Conference on Graph Theory, Combinatorics, Algorithms and Applications, June 1993
- Two survey lectures at East China Normal University, Shanghai, China, June 1993
- Mathematical Sciences Lecture Series, Johns Hopkins University, Baltimore, MD, June 1993
- Colloquium, University of Tennessee, Knoxville, TN, April 1994
- Colloquium, Louisiana State University, April 1994
- Colloquium, Georgia Institute of Technology, May 1994
- Pure Mathematics Seminar, University of Cardiff, UK, June 1994
- Combinatorics Seminar, Oxford University, Oxford, UK, June 1994
- Discrete Mathematics Seminar, University of Grenoble, France, June 1994
- Discrete Mathematics Seminar, University of Paris VI, June 1994
- Logic Seminar, Mathematical Institute of the Czech Academy of Sciences, Prague, Czech Republic, June 1994
- Graph Theory Conference, Oberwolfach, Germany, June 1994
- Graph Embeddings and Maps on Surfaces, Donovaly, Slovakia, August 1994
- Colloquium, Emory University, Atlanta, GA, October 1994
- Kolloquium über Kombinatorik, University of Hamburg, Germany, November 1994
- Colloquium, University of Georgia, Athens, GA, March 1995
- Colloque Arbres, Versailles, France, June 1995
- Third Slovene International Conference on Graph Theory, Bled, Slovenia, June 1995
- Burlington Mathfest, Burlington, VT, August 1995
- Second joint AMS-SMM meeting, Guanajuato, Mexico, December 1995

#### Prize

1994 D. Ray Fulkerson Prize in Discrete Mathematics, awarded by the American Mathematical Society and the Mathematical Programming Society to the paper "Hadwiger's conjecture for  $K_6$ -free graphs."